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Journal of Geometry and Physics 49 (2004) 187–196
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Integration of twisted Poisson structures

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Received 24 February 2003; received in revised form 16 April 2003

Abstract

Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids. Twisted Poisson manifolds considered by Ševera and Weinstein [Prog. Theor. Phys. Suppl. 144 (2001) 145] are a natural generalization of the former which also arises in string theory. In this note it is proved that twisted Poisson manifolds are in bijection with a (possibly singular) twisted version of symplectic groupoids.

MSC: 53D17

JGP SC: Differential geometry

Keywords: Poisson manifolds; Symplectic groupoids

1. Introduction

Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids [6], i.e., Lie groupoids endowed with a multiplicative symplectic form. Up to singularities, Poisson manifolds may be integrated to symplectic groupoids as described in [2] (conditions under which integration with no singularities is possible are given in [4]). In this paper we generalize this result to the case when the two structures (of symplectic groupoid and of Poisson manifold) are twisted by a closed 3-form.

Let M be a smooth manifold. A pair (π, ϕ), where π is a bivector field and ϕ is a closed 3-form, is called a *twisted Poisson structure* if it satisfies the equation

$$[\pi, \pi] = \tfrac{1}{2} \wedge^3 \pi^{\#}\phi, \tag{1.1}$$

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0393-0440/$ – see front matter
doi:10.1016/S0393-0440(03)00086-X

where [,] denotes the Schouten–Nijenhuis bracket and $\pi^\#$ is the vector bundle homomorphism $T^*M \to TM$ induced by π (viz., $\pi^\#(x)(\sigma) := \pi(x)(\sigma, \bullet)$, with $x \in M$, $\sigma \in T^*_x M$). According to [14], one also says that π is a *ϕ-Poisson tensor*. In the case $\phi = 0$ one recovers the usual notions of Poisson tensor and Poisson manifold. Twisted Poisson structures have been extensively studied in the physics literature, e.g. [5,9,11].

As explained in [14], a twisted Poisson structure induces a Lie algebroid structure on T^*M with anchor map $\pi^\#$ and Lie bracket of sections σ and τ defined by

$$[\sigma, \tau] := \mathrm{L}_{\pi^\#\sigma}\tau - \mathrm{L}_{\pi^\#\tau}\sigma - \mathrm{d}\pi(\sigma, \tau) + \phi(\pi^\#\sigma, \pi^\#\tau, \bullet). \tag{1.2}$$

In particular, $\forall f, g \in C^\infty(M)$ we have:

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\} + \phi(X_f, X_g, \bullet), \tag{1.3}$$

and

$$[X_f, X_g] = X_{\{f,g\}} + \pi^\#(\phi(X_f, X_g, \bullet)), \tag{1.4}$$

with $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ and $X_f = \pi^\# \mathrm{d}f$.

We will denote this Lie algebroid by $T^*M_{(\pi,\phi)}$. Sections of its exterior algebra are ordinary differential forms. One may define a derivation δ deforming the de Rham differential by ϕ; viz., we define a graded derivation $\delta : \Omega^*(M) \to \Omega^{*+1}(M)$ by setting $\delta f = \mathrm{d}f$ if $f \in C^\infty(M)$ and

$$\delta\sigma = \mathrm{d}\sigma - \iota_{\pi^\#\sigma}\phi,$$

if $\sigma \in \Omega^1(M)$. It turns out that

$$\delta[\sigma, \tau] = [\delta\sigma, \tau] + [\sigma, \delta\tau], \quad \forall \sigma, \tau \in \Omega^1(M),$$

and that $\delta^2 = [\phi, \bullet]$ (where we have extended the Lie bracket to the whole of $\Omega^*(M)$ as a biderivation). So $(T^*M_{(\pi,\phi)}, \delta)$ constitutes an example of a quasi-Lie bialgebroid [8,12], a generalization of Drinfeld's quasi-Lie bialgebras [7,10].

If $T^*M_{(\pi,\phi)}$ may be integrated to a Lie groupoid $(G \rightrightarrows M, \alpha, \beta)$ (i.e., if it exists a Lie groupoid G whose Lie algebroid is $T^*M_{(\pi,\phi)}$), the differential δ induces extra structure on G. Namely, denoting by α and β the source and target maps of G, then G may be endowed with a non-degenerate, multiplicative 2-form ω that satisfies

$$\mathrm{d}\omega = \alpha^*\phi - \beta^*\phi.$$

In other words, (ω, ϕ) is a 3-cocycle for the double complex $\Omega^*(G^{(*)})$, where $G^{(0)} = M$, $G^{(1)} = G$ and elements of $G^{(k)}$ are k-tuples of elements of G that may be multiplied (in the given order). One differential is de Rham and the other is the groupoid-complex differential. Observe that in the true Poisson case (i.e., $\phi = 0$), ω is closed, so G is an ordinary symplectic groupoid. In the general case, G is called a *non-degenerate twisted symplectic groupoid*, and the non-degenerate 2-form ω is said to be *relatively ϕ-closed*. The main theorem of the paper (conjectured in [14]) is the following one.

Theorem. *There is a bijection between integrable twisted Poisson structures and source-simply connected non-degenerate twisted symplectic groupoids.*

Here "integrable twisted Poisson structure" means that the associated Lie algebroid is integrable.

In Section 2 we give an introduction to non-degenerate twisted symplectic groupoids and prove that they induce twisted Poisson structures on the base manifolds (Theorem 2.6 on page 7).

In Section 5 we prove the theorem, though in a more general setting. In fact, as shown in the generalization [3] (see also [13]) of the construction given in [2]), to any Lie algebroid A one can associate a topological source-simply connected groupoid $G(A)$, which is the Lie groupoid integrating A whenever A is integrable. The topological groupoid $G(A)$ is defined as the leaf space of a smooth foliation, as we recall in Section 3; so it makes sense to define on it a notion of smooth functions and forms. In the case when A is $T^*M_{(\pi,\phi)}$, we prove that $G(A)$ may always be endowed with a non-degenerate, multiplicative, relatively ϕ-closed 2-form ω. The construction is a modification, described in Section 4, of the method developed in [2], where the true Poisson case (i.e., $\phi = 0$) was dealt with.

As a final remark, we mention that general multiplicative 2-forms, their infinitesimal counterparts and their integrations are being treated in [1].

2. Non-degenerate twisted symplectic groupoids

Definition 2.1. A non-degenerate twisted symplectic groupoid is a Lie groupoid $(G \rightrightarrows M, \alpha, \beta)$ equipped with a non-degenerate 2-form $\omega \in \Omega^2(G)$ and a 3-form $\phi \in \Omega^3(M)$ such that:

1. $d\phi = 0$;
2. $d\omega = \alpha^*\phi - \beta^*\phi$;
3. ω is multiplicative, i.e., the 2-form $(\omega, \omega, -\omega)$ vanishes when being restricted to the graph of the groupoid multiplication $\Lambda \subset G \times G \times G$.

Let π_G denote the bivector field on G corresponding to ω. Then (π_G, Ω), where $\Omega = \alpha^*\phi - \beta^*\phi$, defines a twisted Poisson structure on G in the sense of [14].

For any $\xi \in \Gamma(A)$, by $\vec{\xi}$ and $\overleftarrow{\xi}$ we denote its corresponding right and left invariant vector fields on the groupoid G, respectively. The following properties can be easily verified.

Proposition 2.2.

1. $\epsilon^*\omega = 0$, *where* $\epsilon : M \to G$ *is the natural embedding*;
2. $i^*\omega = -\omega$, *where* $i : G \to G$ *is the groupoid inversion*;
3. *for any* $\xi, \eta \in \Gamma(A)$, $\omega(\vec{\xi}, \vec{\eta})$ *is a right invariant function on G, and* $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ *is a left invariant function on G*;
4. $\omega(\vec{\xi}, \overleftarrow{\eta}) = 0$;
5. $\omega(\vec{\xi}, \vec{\eta})(x) = -\omega(\overleftarrow{\xi}, \overleftarrow{\eta})(x^{-1})$.

Proof. The proof is standard, and essentially follows from the multiplicativity of ω:

1. For any $\delta'_m, \delta''_m \in T_mM$, since $(\delta'_m, \delta'_m, \delta'_m), (\delta''_m, \delta''_m, \delta''_m) \in T\Lambda$, it follows that $\omega(\delta'_m, \delta''_m) = 0$.

2. $\forall x \in G$ and $\forall \delta'_x, \delta''_x \in T_x G$, it is clear that $(\delta'_x, i_*\delta'_x, \alpha_*\delta'_x), (\delta''_x, i_*\delta''_x, \alpha_*\delta''_x) \in T\Lambda$. Thus, by (1), we have
$$\omega(\delta'_x, \delta''_x) + \omega(i_*\delta'_x, i_*\delta''_x) = 0,$$
and (2) follows.
3. For any $\xi, \eta \in \Gamma(A)$, $(\vec{\xi}(x), 0_y, \vec{\xi}(xy)), (\vec{\eta}(x), 0_y, \vec{\eta}(xy)) \in T\Lambda$. Thus
$$\omega(\vec{\xi}(x), \vec{\eta}(x)) - \omega(\vec{\xi}(xy), \vec{\eta}(xy)) = 0.$$
Hence $\omega(\vec{\xi}, \vec{\eta})$ is a right invariant function on G. Similarly, $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ is a left invariant function on G.
4. By considering the vectors $(\vec{\xi}(x), 0_{\beta(x)}, \vec{\xi}(x))$ and $(0_x, \overleftarrow{\eta}(\beta(x)), \overleftarrow{\eta}(x)) \in T\Lambda$, we obtain $\omega(\vec{\xi}(x), \overleftarrow{\eta}(x)) = 0$.
5. Follows from (2) and the fact that $i_*\vec{\xi} = -\overleftarrow{\xi}$. □

Define a section $\gamma \in \Gamma(\wedge^2 A^*)$ and a bundle map: $\lambda : A \to T^*M$ by
$$\omega(\vec{\xi}, \vec{\eta}) = \alpha^*\gamma(\xi, \eta), \quad \forall \xi, \eta \in \Gamma(A), \tag{2.1}$$
and
$$\langle \lambda(\xi), v\rangle = \omega(\vec{\xi}(m), v), \quad \forall \xi \in A|_m, v \in T_m M. \tag{2.2}$$

Lemma 2.3.

1. $\omega(\overleftarrow{\xi}, \overleftarrow{\eta}) = -\beta^*\gamma(\xi, \eta), \forall \xi, \eta \in \Gamma(A)$;
2. *for all* $\xi, \eta \in \Gamma(A)$,
$$\gamma(\xi, \eta) = \langle \rho(\xi), \lambda(\eta)\rangle; \tag{2.3}$$
3. $\lambda : A \to T^*M$ *is a vector bundle isomorphism.*

Proof.

1. Follows from Proposition 2.2 (5).
2. We have
$$\omega(\vec{\xi}, \vec{\eta}) = \omega(\vec{\xi} - \overleftarrow{\xi}, \vec{\eta}) = \omega(\vec{\eta}, \rho(\xi)) = \langle \rho(\xi), \lambda(\eta)\rangle.$$
3. Assume that $\lambda(\xi) = 0$. That is, $\omega(\vec{\xi}(m), v) = 0, \forall v \in T_m M$, which implies that $\vec{\xi}(m)\omega = 0$ by Proposition 2.2 (4). Hence $\xi = 0$ since ω is non-degenerate. This means that λ is injective. On the other hand, assume that $v \in (\lambda(A|_m))^\perp$. Then $\omega(\vec{\xi}(m), v) = 0, \forall \xi \in A|_m$. Thus $v\omega = 0$ using Proposition 2.2 (1), which implies that $v = 0$. Therefore λ is surjective. □

Lemma 2.4. *For any* $f \in C^\infty(M)$
$$\overrightarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\alpha^* f}; \qquad \overleftarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\beta^* f}. \tag{2.4}$$

Proof. First, one shows that $X_{\alpha^* f}$ is a right invariant vector field on G and $X_{\beta^* f}$ is a left invariant vector field. This can be shown using the same argument as in the case of symplectic groupoids [6]. Namely the multiplicativity of ω together with dimension counting implies that the graph Λ is coisotropic with respect to $(\pi_G, \pi_G, -\pi_G)$. The later implies that $X_{\alpha^* f}$ is a right invariant vector field on G and $X_{\beta^* f}$ is a left invariant vector field.

Second, for any $v \in T_m M$, we have

$$\omega(X_{\alpha^* f}(m), v) = \langle \alpha^* \mathrm{d}f(m), v\rangle = \langle \mathrm{d}f(m), \alpha_* v\rangle = \langle \mathrm{d}f(m), v\rangle.$$

It thus follows that $\lambda(X_{\alpha^* f}) = \mathrm{d}f$, or $\overrightarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\alpha^* f}$. The other equation can be proved similarly. □

By pulling back the 2-form $\gamma \in \Gamma(\wedge^2 A^*)$ via λ^{-1}, one obtains a bivector field $\pi \in \Gamma(\wedge^2 TM)$. We introduce a bracket and Hamiltonian vector fields by the usual definitions, i.e., $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ and $X_f = \pi^\#(\mathrm{d}f)$.

Corollary 2.5.

$$\alpha_*\pi_G = \pi; \qquad \beta_*\pi_G = -\pi; \tag{2.5}$$

or equivalently

$$\alpha_* X_{\alpha^* f} = X_f; \qquad \beta_* X_{\beta^* f} = -X_f, \ \ \forall f \in C^\infty(M). \tag{2.6}$$

Proof. For any $f, g \in C^\infty(M)$,

$$\{\alpha^* f, \alpha^* g\} = \omega(X_{\alpha^* f}, X_{\alpha^* g}) = \omega(\overrightarrow{\lambda^{-1}(\mathrm{d}f)}, \overrightarrow{\lambda^{-1}(\mathrm{d}g)}) = \alpha^*(\pi(\mathrm{d}f, \mathrm{d}g)) = \alpha^*\{f, g\}.$$

Similarly, we have $\{\beta^* f, \beta^* g\} = -\beta^*\{f, g\}$. □

We are now ready to prove the main result of the section.

Theorem 2.6.

1. *π is a ϕ-Poisson tensor in the sense of [14], i.e., it satisfies (1.1).*
2. *The bundle map $\lambda : A \to T^*M$ establishes a Lie algebroid isomorphism, where the Lie algebroid on T^*M is induced by the twisted Poisson tensor π as given by Eq. (1.2).*

Proof. Let $\Omega = \alpha^*\phi - \beta^*\phi$. Thus $\forall f, g \in C^\infty(M)$

$$(X_{\alpha^* f} \wedge X_{\alpha^* g})\Omega = (X_{\alpha^* f} \wedge X_{\alpha^* g})\alpha^*\phi = \alpha^*[(\alpha_* X_{\alpha^* f} \wedge \alpha_* X_{\alpha^* g})\phi] = \alpha^*[X_f \wedge X_g \phi].$$

Thus by Eq. (1.4)

$$[X_{\alpha^* f}, X_{\alpha^* g}] - X_{\{\alpha^* f, \alpha^* g\}} = \pi_G^\#(\Omega(X_{\alpha^* f}, X_{\alpha^* g}, \bullet) = \pi_G^\#(\alpha^*\phi(X_f, X_g, \bullet)).$$

Thus it follows that

$$\lambda[X_{\alpha^* f}, X_{\alpha^* g}] = \mathrm{d}\{f, g\} + \phi(X_f, X_g, \bullet).$$

Note that λ intertwines the anchors: $\pi^\# \circ \lambda = \rho$, according to Eq. (2.3). Therefore, using Lie algebroid properties, one shows that the push forward Lie algebroid on T^*M via λ is given by Eq. (1.2). This forces, by the Jacobi identity, π to be ϕ-Poisson, and λ is a Lie algebroid isomorphism between A and $(T^*M)_{\pi,\phi}$. □

3. Integration of $T^*M_{(\pi,\phi)}$

We briefly describe the integration procedure for Lie algebroids of [3,13], adapted to the case of $T^*M_{(\pi,\phi)}$. First one defines the manifold $P(T^*M_{(\pi,\phi)})$ of C^1-Lie algebroid morphisms $TI \to T^*M_{(\pi,\phi)}$, where I is the interval [0, 1] and TI is given its canonical Lie algebroid structure. An element of $PT^*M_{(\pi,\phi)}$ consists of a C^2-path $X : I \to M$ together with a section η of $T^*I \otimes X^*T^*M$ satisfying

$$\mathrm{d}X = \pi^\#(X)\eta.$$

On this manifold one may consider as equivalent two elements which are related by a Lie algebroid morphism $T(I \times I) \to T^*M_{(\pi,\phi)}$ that fixes the endpoints. The quotient space $G(T^*M_{(\pi,\phi)})$ may be given a groupoid structure. For our purposes it is however better to use a different description of $G(T^*M_{(\pi,\phi)})$, i.e., as the leaf space of a foliation. Namely, let $P_0\Gamma(T^*M_{(\pi,\phi)})$ be the space of C^2-paths in the Lie algebra of sections of $T^*M_{(\pi,\phi)}$ with endpoints at zero. We give this space the structure of a Lie algebra by the pointwise Lie bracket. One may then define an infinitesimal action of this Lie algebra on $P(T^*M_{(\pi,\phi)})$. To describe it, we prefer to introduce local coordinates $\{x^i\}$ on M (alternatively, one may use a torsion-free connection). Since $\{\mathrm{d}x^i\}$ is a local basis of sections of $T^*M_{(\pi,\phi)}$, we may define structure functions f by

$$[\mathrm{d}x^i, \mathrm{d}x^j] = f_k^{ij}\,\mathrm{d}x^k,$$

where a sum over repeated indices is understood. If we write locally $\pi = \pi^{ij}\partial_i\partial_j$ and $\phi = \phi_{ijk}\,\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k$, we may compute:

$$f_k^{ij} = \partial_k\pi^{ij} + \pi^{mi}\pi^{nj}\phi_{mnk}.$$

The action is then as follows. To $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$ we associate a vector field ξ_B on $P(T^*M_{(\pi,\phi)})$. We can always write $\xi_B = \xi_B^h + \xi_B^v$ with $\xi_B^h(X,\eta) \in \Gamma(I, X^*TM)$ and $\xi_B^v(X,\eta) \in \Gamma(I, T^*I \otimes X^*T^*M)$. We set then

$$(\xi_B^h(X,\eta))^i = -\pi^{ij}(X)\,(B_X)_j, \tag{3.1a}$$

$$(\xi_B^v(X,\eta))_i = -\mathrm{d}(B_X)_i - f_i^{rs}(X)\,\eta_r\,(B_X)_s, \tag{3.1b}$$

where B_X is the section of X^*T^*M defined by $B_X(t) = B(t)(X(t))$.

Thus, the infinitesimal action of $P_0\Gamma(T^*M_{(\pi,\phi)})$ defines a foliation on $P(T^*M_{(\pi,\phi)})$ and $G(T^*M_{(\pi,\phi)})$ is its quotient space. Let us briefly recall its groupoid structure. The target map α associates to a class of morphisms (X,η) the value of X at 0, while the source map β associates it to the value of X at 1 (observe that the infinitesimal action preserves the endpoints of X). The identity section associates to a point m in M the class $\epsilon(m)$ of the constant path at m with $\eta = 0$. The product is obtained by joining the base paths and restricting the fiber maps consequently (the product is more precisely defined on smooth representatives such that η vanishes with its derivatives at the endpoints).

4. Quasi-symplectic reduction

In this section we describe how to obtain $G(T^*M_{(\pi,\phi)})$ by some sort of symplectic reduction, though our replacement for a symplectic form will be a non-degenerate but not necessarily closed 2-form.

Let T^*PM denote the manifold of C^1-bundle maps $TI \to T^*M$ (over C^2-maps). This space is morally a cotangent bundle and as such it has a canonical symplectic structure Ω_0. Explicitly, a point in T^*PM is a pair (X, η), where X is a C^2-path $I \to M$ and η is a C^1-section of $T^*I \otimes X^*T^*M$. The tangent space at (X, η) is the direct sum of $T^h_{(X,\eta)}T^*PM = \Gamma(I, X^*TM)$ and $T^v_{(X,\eta)}T^*PM = \Gamma(I, T^*I \otimes X^*T^*M)$. Using this splitting, we write

$$\Omega_0(X, \eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \int_I \langle e_1, \xi_2\rangle - \langle e_2, \xi_1\rangle, \tag{4.1}$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between tangent and cotangent fibers of M.

Using the 3-form ϕ on M we may also define a second 2-form on T^*PM:

$$\Omega_1(X, \eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \frac{1}{2}\int_I \phi(X)(\pi^\#(X)\eta, \xi_1, \xi_2). \tag{4.2}$$

The 2-form $\Omega = \Omega_0 + \Omega_1$ is still non-degenerate but no longer closed.

The manifold $P(T^*M_{(\pi,\phi)})$ introduced in the previous section may be regarded as a submanifold of T^*PM. If we introduce "momentum maps" $H : T^*PM \to P_0\Gamma(T^*M_{(\pi,\phi)})^*$ by

$$H_B(X, \eta) = \int_I \langle B_X, \mathrm{d}X - \pi^\#(X)\eta\rangle,$$

then $P(T^*M_{(\pi,\phi)})$ is $H^{-1}(0)$. One may check that $\mathrm{d}H_B$ lies in the image of Ω for any $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$; so, since Ω is non-degenerate, one may define a map $B \to \hat{\xi}_B$ that associates a vector field $\hat{\xi}_B$ on T^*PM to B by

$$\iota_{\hat{\xi}_B}\Omega = \mathrm{d}H_B. \tag{4.3}$$

One may easily check that the restriction of $\hat{\xi}_B$ to $P(T^*M_{(\pi,\phi)})$ is tangent to it. More to the point, one may check that the vector field on $P(T^*M_{(\pi,\phi)})$ so obtained is precisely the ξ_B of (3.1) which defines the infinitesimal action of $P_0\Gamma(T^*M_{(\pi,\phi)})$ on $P(T^*M_{(\pi,\phi)})$.

5. Proof of the theorem

In the setting of the previous section, we want to prove that the restriction Ω of Ω to $P(T^*M_{(\pi,\phi)})$ is basic w.r.t. to the projection $p : P(T^*M_{(\pi,\phi)}) \to G(T^*M_{(\pi,\phi)})$, viz., $\Omega = p^*\omega$; moreover, we want to prove that ω satisfies all the required conditions.

Observe that Ω is automatically horizontal by (4.3). On the other hand, unlike the usual symplectic case, it is not clear that Ω is also invariant; in fact, at first, we may only see that $\mathrm{L}_{\xi_B}\Omega = \iota_{\xi_B}\,\mathrm{d}\Omega = \iota_{\xi_B}\,\mathrm{d}\Omega_1$, where Ω_1 denotes the restriction of Ω_1 to $P(T^*M_{(\pi,\phi)})$. To proceed, we must understand Ω_1 better.

Let PM be the manifold of C^2-paths in M. Let $\mathrm{ev} : I \times PM \to M$ be the evaluation map and $\mathrm{pr} : I \times PM \to PM$ the projection to the second factor. Define $\Phi = \mathrm{pr}_* \mathrm{ev}^* \phi \in \Omega^2(PM)$, where pr_* denotes integration along the fiber. If we finally denote by $q : P(T^*M_{(\pi,\phi)}) \to PM$ the map that retains only the base map of the Lie algebroid morphism, we realize immediately that

$$\Omega_1 = q^*\Phi.$$

By the generalized Stokes' theorem and the fact that ϕ is closed, we obtain $\mathrm{d}\Phi = \underline{\alpha}^*\phi - \underline{\beta}^*\phi$, where $\underline{\alpha}$ and $\underline{\beta}$ are the maps $PM \to M$ that assign to a path its values at 0 and 1, respectively. Thus

$$\mathrm{d}\Omega = q^*(\underline{\alpha}^*\phi - \underline{\beta}^*\phi).$$

Since the vector field ξ_B does not move the endpoints, we conclude that $\iota_{\xi_B}\,\mathrm{d}\Omega = 0$, viz., that Ω is invariant as well. We write then $\Omega = p^*\omega$ as at the beginning of the section. The 2-form ω on $G(T^*M_{(\pi,\phi)})$ is clearly multiplicative since the product is defined by joining the paths and Ω is defined as an integral. Moreover, recalling the definition of the source and target map β and α, we observe that $\underline{\alpha} \circ q = \alpha \circ p$ and $\underline{\beta} \circ q = \beta \circ p$. So we may write the equation above as

$$\mathrm{d}\Omega = p^*(\alpha^*\phi - \beta^*\phi).$$

Since $\mathrm{d}\Omega = p^*\,\mathrm{d}\omega$ and p is a surjection, this shows that ω is relatively ϕ-closed.

Finally, we need to prove that the 2-form ω is non-degenerate. It is clear from the construction that ω is non-degenerate along the identity M. The claim thus follows from the following lemma.

Lemma 5.1. *A multiplicative 2-form $\omega \in \Omega^2(G)$ on a Lie groupoid $G \rightrightarrows M$ is non-degenerate if and only if it is non-degenerate along the identity M.*

Proof. First of all, note that for any $\delta_x \in T_xG$, and $\xi \in \Gamma(A)$, we have

$$\omega(\overleftarrow{\xi}(x), \delta_x) = \omega(\overleftarrow{\xi}(v), \beta_*\delta_x), \tag{5.1}$$

$$\omega(\vec{\xi}(x), \delta_x) = \omega(\vec{\xi}(u), \alpha_*\delta_x), \tag{5.2}$$

where $u = \alpha(x)$ and $v = \beta(x)$. Eq. (5.1), for instance, follows from the fact that both $(\delta_x, \delta_x, \beta_*\delta_x)$, and $(0, \overleftarrow{\xi}(x), \overleftarrow{\xi}(v))$ are tangent to the graph of the groupoid multiplication $\Lambda \subset G \times G \times G$. Eq. (5.2) can be proved similarly. Now assume that $\delta_x \in \ker\omega_x$. It follows from Eq. (5.1) that $\beta_*\delta_x \in \ker\omega_v$ since M is isotropic with respect to ω. Therefore $\beta_*\delta_x = 0$ by assumption. Hence $\delta_x = \vec{\eta}(x)$. On the other hand, according to Eq. (5.2), one has $\omega(\vec{\eta}(u), T_uM) = 0$ since α is a submersion. This implies that $\vec{\eta}(u) \in \ker\omega_u$. Therefore $\vec{\eta}(u) = 0$ by assumption. This implies that $\delta_x = \vec{\eta}(x) = 0$. This concludes the proof. □

We need now to prove that the correspondence between ϕ-twisted Poisson structures and twisted symplectic groupoids is a bijection. The proof is divided into two steps:

Step 1. By construction (see [2,3]) the Lie algebroid of $G(T^*M_{(\pi,\phi)})$ is $T^*M_{(\pi,\phi)}$. As discussed in Section 2, the relatively ϕ-closed, multiplicative, non-degenerate 2-form ω determines an automorphism λ of T^*M and a bivector field γ on M as in Eqs. (2.1) and (2.2). We have to show that λ is the identity and that $\gamma = \pi$. First of all we observe that it is enough to consider (2.1) at the unit element $\epsilon(m) \in G(T^*M_{(\pi,\phi)})$ corresponding to $m \in M$:

$$\omega(\epsilon(m))(\vec{\xi_1}(\epsilon(m)), \vec{\xi_2}(\epsilon(m))) = \gamma(m)(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in A|_m.$$

By construction $\epsilon(m)$ is the equivalence class of the path $X(t) = m$, $\eta(t) = 0$, $\forall t \in I = [0, 1]$. The vector field $\vec{\xi_i}$, $i = 1, 2$, evaluated at $\epsilon(m)$ is the projection to $T_{\epsilon(m)}G(T^*M_{(\pi,\phi)})$ of the vector $\hat{\xi}_i \in T_{(m,0)}P(T^*M_{(\pi,\phi)})$ defined by $\hat{\xi}_i(t) = (\pi^\#(m)\xi_i\, t, \xi_i\, \mathrm{d}t)$. Observing then that for $\eta = 0$ the 2-form Ω_1 of Eq. (4.2) vanishes, we get, also using (4.1)

$$\omega(\epsilon(m))(\vec{\xi_1}(\epsilon(m)), \vec{\xi_2}(\epsilon(m))) = \Omega_0(m, 0)(\hat{\xi}_1, \hat{\xi}_2) = 2\int_0^1 \pi(m)(\xi_1, \xi_2)t\, \mathrm{d}t$$
$$= \pi(m)(\xi_1, \xi_2),$$

which shows $\gamma = \pi$. As for (2.2), observe that $\omega(\epsilon(m))(\vec{\xi_1}(\epsilon(m)), v)$ is just $\Omega_0(m, 0)(\hat{\xi}_1, \hat{v})$ with $\hat{v}(t) = (v, 0)$. As a consequence

$$\omega(\epsilon(m))(\vec{\xi_1}(\epsilon(m)), v) = \int_0^1 \langle \xi_1, v\rangle\, \mathrm{d}t = \langle \xi_1, v\rangle,$$

which shows that λ is the identity.

Step 2. Assume that $(G \rightrightarrows M, \omega + \phi)$ is an α-simply connected non-degenerate twisted symplectic groupoid. Let π be its induced ϕ-twisted Poisson structure on M. Then the above integration process integrates the Lie algebroid $T^*M_{(\pi,\phi)}$ into a Lie groupoid, which is known to be isomorphic to $G \rightrightarrows M$, and a multiplicative 2-form ω' on that groupoid. By identifying this groupoid with $G \rightrightarrows M$, therefore one may think ω' as a multiplicative 2-form on G. One needs to show that $\omega' = \omega$. By Step 1, we conclude that ω' and ω must coincide along the identity space M. Let $\tilde{\omega} = \omega - \omega'$. Then $\tilde{\omega}$ is a multiplicative closed 2-form on G and $\tilde{\omega}|_M = 0$. Given any $\xi \in \Gamma(A)$, it is easy to see that $(\vec{\xi}(\alpha(x)), 0, \vec{\xi}(x))$ is tangent to the graph Λ of groupoid multiplication. On the other hand, for any $\delta_x \in T_xG$, it is also clear that $(\alpha_*\delta_x, \delta_x, \delta_x) \in T\Lambda$. It thus follows that

$$\tilde{\omega}(\vec{\xi}(\alpha(x)), \alpha_*\delta_x) - \tilde{\omega}(\vec{\xi}(x), \delta_x) = 0.$$

Therefore we have $\vec{\xi}\tilde{\omega} = 0$. Thus

$$L_{\vec{\xi}}\tilde{\omega} = (di_{\vec{\xi}} + i_{\vec{\xi}}d)\tilde{\omega} = 0,$$

which implies that $\tilde{\omega} = 0$ since any point in G can be reached by a product of (local) bisections generated by $\vec{\xi}$. This concludes the proof.

Acknowledgements

We thank Jim Stasheff and Alan Weinstein for useful discussions and comments. We acknowledge Zürich University (PX) and Penn State University (ASC) for their kind hospitality during the preparation of the work. ASC acknowledges partial support of SNF Grant No. 20-63821.00. PX acknowledges partial support of NSF Grant DMS00-72171.

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